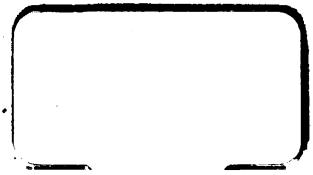

ON THE THEORY OF THE INFINITE IN MODERN THOUGHT

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TWO INTRODUCTORY STUDIES

BY

E. F. JOURDAIN

DOCTOR OF THE UNIVERSITY OF PARIS

VICE-PRINCIPAL, ST. HUGH'S HALL, OXFORD

LONGMANS, GREEN AND CO.

39 PATERNOSTER ROW, LONDON

NEW YORK, BOMBAY, AND CALCUTTA

1911

Of the two papers here reproduced, the first was given in 1905 to a meeting of the women science students in Oxford; the second, in 1908, to the Philosophical Society of this College. They are printed by request, with the author's apologies for their incompleteness. The lecture form has been retained. I am indebted to my brother, Mr. P. Jourdain, for help in preparing the first lecture, and for his revision of the text.

E. F. JOURDAIN.

St. Hugh's Hall, Oxford,
January, 1911.

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I

THE PROBLEM OF THE FINITE AND THE INFINITE

THE influence of mathematics on philosophy and vice versâ can be inferred from the historical progress of both studies, though it has not been possible till about within the last fifteen years to give a logical explanation for the relations between them. As long as it was believed, according to the Kantian view, that the science of mathematics was based on intuitions of time and space, the alliance between philosophy and mathematics could not be proved to be closer than that between philosophy and experimental science, although the historical fact remained that philosophy and mathematics exercised a mutual stimulus, and developed at the same periods of history.

But mathematics, as now defined, is inde-

pendent of intuitions of space and time, and also of axioms and hypotheses.[1] Mathematics, as now understood, is based, like formal logic, on the prerequisites of thought, not on the notions of space and time. Here there is no definition of number or space, but the conception of number and space,[2] which is more complicated, can be derived from them. All other complicated mind processes can, in the same way, be reduced to the simple elements of the prerequisites of thought.

Such a science might exist out of conditions of time and space as we know them. It is a science of relations rather than of mere number. Founded, then, on the laws of symbolic logic, it

[1] Of course, if we comprehend in our view only elementary geometrical and algebraical science, it is easy to show that they *do* demand both axioms and intuitions. Take, *e.g.* Euclid I. 1., where in the construction it is necessary to employ intuition for the assertion that the arcs really cut one another. There is no logical certainty that they do; in fact, in some other conditions, *e.g.* in those of other space dimensions, they might not.

[2] This is, of course, not the space of experience. Logic and mathematics deal with implications of thought. See B. Russell (*Hibbert Journal*, 1904, pp. 809–12), who has shown that in all pure mathematics it is only the implications that are asserted, not the premiss or the consequence, as mathematicians used formerly to assume.

is a valuable aid and illustration to philosophy; philosophy, on the other hand, can imagine lines for the exercise of the constructive power involved in mathematics. It is the object of this paper to show that the close though apparently accidental union of philosophy and mathematics throughout the history of thought can now be explained, and that the problems with which pure mathematics is now concerned are those which lie at the core of philosophic thought and speculation. (Symbolic logic has developed to meet the new demands made upon it. It does not now reduce itself to the syllogism, as Aristotle thought it did; the prerequisites of thought are shown to be manifold instead of single.[1])

The use of the word philosophic in this connection suggests a necessity for further definition. Philosophy is held to include at least two great branches—Metaphysics and Ethics. The influence of mathematics is most evident on the metaphysical side of philosophy; in fact, the grouping of mathe-

[1] De Morgan, Peirce, Schröder, and B. Russell have worked out the logic of relations as well as the syllogism.

matics and metaphysics as allied sciences tends to bring out the essential distinction between metaphysics and ethics, and—though not by any means to imply a break in their real relation—to show where this has been misunderstood. No philosophy has been equally strong on both sides; they represent different forms of activity of the human mind; but it is still true, and from the conditions always must be, that an ethical system grows out of metaphysics as practice follows precept and conduct implies belief. The new definition of mathematics does not touch these consequences; it merely marks the limits within which philosophy on the metaphysical side can submit to, or rest upon, the conclusions of mathematics.

As to the historical relation between metaphysics and mathematics, the subject is so vast that we shall only attempt a very rapid generalisation of its results on the growth of the conception of the Finite and the Infinite. (Of course, there are many other sides of the relation which might be studied.) The general result of the inquiry has been, as far as we

can judge, that metaphysics has exercised an inspiring force on mathematics, and mathematics has defined and strengthened the conceptions of metaphysics at every critical stage in the history of philosophy. But where metaphysics has been treated as the proof of science, where it has been laid down as the foundation for exact knowledge, the results have not corresponded with the truth of experience, and the quality of thought has become degenerate. Progress depends on the right perception of the relations between the sciences and parts of philosophy.

Such progress is especially evident in the early Greek and in the modern periods, while the large period from the Christian era to the Renaissance gives examples of the unfortunate reversal of the parts of metaphysics and science and consequent confusion of thought.[1]

The problems of the metaphysician are no doubt in a sense always the same; but this is equally true of the problems of any other science. The methods by which the problems are attacked and the adequacy of the solutions

[1] See Taylor, "Elements of Metaphysics," p. 13.

they receive vary, from age to age, in close correspondence with the general development of science. Every great metaphysical conception has exercised its influence on the general history of science, and in return every important movement in science has affected the development of metaphysics. The metaphysician could not if he would, and would not if he could, escape the duty of estimating the bearing of the great scientific theories of his time upon our ultimate conceptions of the nature of the world as a whole. Every fundamental advance in science thus calls for a restatement and reconsideration of the old metaphysical problems in the light of the new discovery.

During the Greek period mathematics was the only branch of science which was at all developed, and its development coincided with the age of the philosophers. Thus when Plato spoke of science he always meant mathematics. And even later, when the physical sciences had begun to develop, Aristotle put mathematical ideas into close connection with

metaphysical ones when he stated that they occupied the middle term between the ideal and the sensible. Both Plato and Aristotle referred to and depended upon mathematical proofs and illustrations of philosophical questions. During this Greek period the conception of Infinity took shape. The pre-Platonic notion, reproduced again later in the decline of Socratic theory by the Stoics, was that the Infinite was the aggregate of the Finite; the Platonic and Aristotelian theory, that, namely, of the most vigorous moment of Greek thought, was that the Infinite was *more* than the aggregate of the Finite; that it had a self-determined existence from which the Finite had been derived. Existence, as known to man, was treated as a compromise between the Finite and the Infinite.

Neo-Platonism altogether separated the Infinite from the Finite. In the Alexandrine metaphysics, which represented a decadent stage of philosophy and its deviation from the sciences, the conception of the Infinite became less clear and logical; it diverged from the view which had been affected by mathematical

thought, and tended to assimilate to itself the ideas of perfection and universality, which, philosophically speaking, are conceptions distinct from that of Infinity—universality referring to a common principle of unity, and perfection involving the moral ideal. Real progress was deferred by the too rapid coherence of ideas only partially analysed and understood. Thinkers passed quickly from the exclusive contemplation of subject to that of object and back again,[1] each new period negativing all previous experience, till the result was the exclusion of an imperfectly analysed Relative and Finite from an insufficiently apprehended Absolute and Infinite.

After the Christian era Greek philosophy drifted off into scholasticism and lost touch of reality, the grammar of Aristotelian logic replacing the vital connection of ideas. St. Anselm, it is true, attempted to find a rational proof of the existence of God, and identified Him with the Infinite of Greek thought; but St. Thomas Aquinas led away the argument

[1] See Dr. Caird, "Evolution of Theology in the Greek Philosophers."

to a discussion as to how far form and matter, separately considered, shared in the quality of Infinity. (He thought form did, but not matter.) An overpowering sense of mystery, joined to a premature desire for definition without scientific analysis, sapped the vigour of mediæval thought.

Throughout the middle ages, then, we see the conditions of the Greek period reversed: philosophy during the second period is not, as in the first, engaged in giving a stimulus to the efforts of pure reason; rather the intuitions of philosophy are treated as axiomatic, and a false superstructure of knowledge, alien to experience and reality, is erected upon these foundations. Philosophy, in fact, is used as a general basis for science. The parts of philosophy and mathematics, correctly though imperfectly seen by the Greeks, are in the second period exchanged, and the result is confusion of idea. The notion of the Infinite, as in the Alexandrine metaphysics, is held to include perfection and universality, and does not exist as a conception apart from these.

After the Renaissance, the scholastic philosophy falling into disuse, the attempt to find an explanation of the Cosmos, a synthesis of the universe, was abandoned, and replaced by the Cartesian idea—the inference of existence from thought, and the limitation of the sphere of inquiry to that which could be known by the ego. New scientific and mathematical discoveries kept pace with this new analysis and development of thought,[1] and the surer ground in philosophy was definitely allied with the work of the mathematical mind. The philosophical thesis developed from "The Infinite is the negation of the Finite," to "The Infinite presupposes the Finite and does not exclude it." The problem of the Finite and the Infinite became the great idea of the age, and there was a reversion to the Greek notion of existence as a compromise between the two, and almost the hint of a coming explanation of them. In the decline of Cartesian philosophy, when it drifted off into Pantheism, there was

[1] So Galileo, Newton, Huygens were philosophers in science. Descartes, Pascal, Leibniz were mathematicians as well as philosophers.

only a vague conception of the Infinite, and we trace a tendency to identify the notion of Infinity with that of the Cosmos. In mediæval thought the idea of the Infinite had become confused with that of the Perfect and Universal; in the modern period the effort to give a concrete expression to the notions of Infinity, Perfection, and Universality diverted the ideas from their relation to the Creator and applied them to the Creation.

Kant, who gave a new impulse to some parts of the Cartesian idea, neglected both mathematical proofs and the search for a metaphysical Absolute. In avoiding the subject he helped to perpetuate the vague descriptions of the Finite and the Infinite, uncorrected by mathematical thought, which had been the currency of the philosophy of his age, and which corrupted the philosophy of the succeeding century. The nineteenth century produced nothing more than guesses at truth, which were, perhaps, not very far wrong, and which the present century is engaged in correcting and substantiating. The same vague-

ness afflicted both mathematics (theory of functions) and philosophy. Fichte, Schelling, and Hegel, particularly Hegel, identify the metaphysical Absolute with reality, infinity, and the universal. The ideas of continuity and infinity are not separated by them from those of perfection and universality, nor from one another, and their nature is not understood.

Leaving aside the French neo-critical School (Renouvier) and the English School (Spencer) —the first of whom deny the Infinite, thus acting in opposition to mathematical reasoning, while the second perpetuate Kant's error of considering the Infinite, though thinkable, as unknowable (Dr. Caird has pointed out that this position is illogical)—we arrive at a moment in history which is more fruitful in result on the mathematical side, and will, no doubt, have an effect on metaphysics. For, owing to recent discoveries in Germany and England, mathematics is now in a position to give greater support than before to the intuitions of philosophy. Hitherto, philosophers have been reluctant to allow full value

to the mathematical conceptions of Infinity, and with some justice, as the notion had not been sufficiently analysed. Philosophers, who never attempted the analysis, have been inclined to accept certain contradictions in their conception as inherent in the nature of Infinity. Within the last twenty-five years Cantor and Dedekind have cleared up the notion of continuity, and Russell has given greater precision to the idea, and has applied this reasoning to philosophy.

Present-day metaphysicians seem to be divided into two groups; on the one side, those who consider in philosophy the value of a theory of being, and, on the other, those who chiefly consider the value of a theory of knowledge, *i.e.* the Epistemologists. The first group, devoting themselves to psychology, evolution, and history, have no *necessary* belief in the Infinite. The Epistemologists, whose work is founded on Kant, discuss the theory of knowledge and enumerate the conditions of knowledge. Their argument may not touch, but does not exclude, the notion of the Infinite. The position of the Epistemologist

has been made infinitely more secure by recent mathematical work. That of the psychologist remains almost untouched. It is necessary now to examine more closely the mathematical results to which reference has been made.

In general terms it may be said that mathematics has, as a study, led immediately from the nature of the subject to the perception of the Infinite, and to a knowledge of the connection between the Infinite and the Finite. The simplest form in which the idea can be put is stated by St. Augustine, who said that numbers considered individually were finite, but considered as an aggregate were infinite.[1] Before St. Augustine, and after him down the long stream of philosophic thought, the theologian and the philosopher have turned to mathematics for illustrations of the infinitely great and infinitely little, as developed from the concrete processes of arithmetic and

[1] See S. Augustine, *De Civitate Dei*, Book XII. ch. xix.: "Ita vero suis quisque numerus proprietatibus terminatur, ut nullus eorum par esse cuicumque alteri possit. Ergo et dispares inter se atque diversi sunt, et singuli quique finiti sunt, et omnes infiniti sunt."

geometry. The recurring decimal in arithmetic, the properties of the circle and ellipse in geometry, of the cone in conic sections, and of the surd in algebra, all touch the problem of number and space on the side of Infinity.

In higher mathematics it is possible to start from the idea of the Finite and reach the conception of the Infinite; or to reverse the process, and from the Infinite to deduce the Finite. Thus in the familiar puzzle of the subdivision of the parts of a straight line by halving the remainder, there will be a crowding and a coalescing of the points of division towards one end of the line, the points of division getting infinitely nearer, but the steps will never meet. Here in the centre of a straight line—a limited straight line—we are confronted with the problem of Infinity.

Again, from a series of finite numbers we can gain the notion of an infinite series. Take two series which have a correspondence with one another. If for every element of the one we can choose an element of the other, and of the other there is an element for the one, when

at any point we cut off its progress to infinity, this happens:—

One series, if summed up, will give a larger numerical result than the other, and therefore can be said to be greater than the second. Let us call the first series A, and the second B. Let us now imagine the two series, though starting at a definite point, are never cut at the further end. Then to all infinity series B is without certain numbers which series A possesses, *and as an infinite series* is smaller than series A. But, on the other hand, when neither series is cut, series B retains its correspondence with series A. Thus we attain a definition of an infinite series. It is such that the part, while being less than the whole, has yet a complete correspondence with the whole. The whole is greater than the part, but take away the part from the whole and that which remains corresponds to it *in infinity*, because the test of summing the series (which would give a contrary result) involves limitation, and thus cannot be applied. Subtraction can take place in Infinity without loss.

By reversing this process, and by starting from the theory of the Infinite, we may gain some idea of the discovery of the Finite. So Dedekind and Russell define finite numbers not only in the usual way as those which can be reached by mathematical induction, starting from 0 and increasing by 1 at each step, but also as those of classes which are *not* similar to the parts of themselves obtained by taking away single terms. That is the reversal of the process applied just now. Dedekind also has deduced the Finite from the Infinite by a novel process. He predicates a world of thought which we each and all possess, filled with thoughts and things, to each thing corresponding a thought. There are thus two "trans-finite" series in the minds of each and all of us; we cannot say when the series of thoughts and things will end; but they have number, though it is infinite number. (Number exists wherever there is a correspondence, one to one, between two aggregates.) But in this *Gedankenwelt*, says Dedekind, there is one thing to which there is no corresponding thought: that is the ego. Each man is part

of his own world of thought, but there is no thought of himself in his mind corresponding exactly to himself, as a thought in his mind corresponds to another object.[1] Two important results follow from Dedekind's theory: first, the existence of a finite number one, the number of the ego, as deduced from the *Gedankenwelt* of two infinite systems; second, by putting together all the *Gedankenwelts* there are or may be, we get the notion of series of series, which seems to transcend Infinity, and it gives us the conditions which are possibly gathered up in the Absolute. Now the argument from the Finite to the Infinite and the converse process may both be employed in mathematics (or both may be neglected, as in the elementary methods of calculation used in arithmetic). A discussion has taken place in the *Hibbert Journal* on the relative value of the two methods. Keyser, in an article called the Axiom of Infinity, argued that one method, that of Dedekind, should be exclusively developed. Russell answered him, stating that

[1] See R. Dekekind, *Was sind und was sollen die Zahlen?* 1893.

it was not necessary to hold exclusively to either. If the Finite and the Infinite can in turn be deduced from one another, neither conception can be truly called an axiom. The real axiom is existence, which includes both, and which is defined by mathematicians as that which is *not self-contradictory*.

Now the problem of Infinity includes also that of continuity; in other words, the problem of number includes that of *cardinal* and *ordinal* number. It is time to get to the mathematical definition of number, which we have found as a conception can be attached both to the Finite and to the Infinite. What is number in mathematics?

Take any collection of things—we call that an *aggregate*. If an aggregate corresponds one to one with another aggregate, they are both said to have a number, and the same number. Subtract from the idea of an aggregate the idea of quality or kind, and order or arrangement, what is left is its *cardinal number*. If you subtract quality and not order, the result is an *ordinal number*. This reasoning applies both to finite and infinite aggregates; in fact,

the Infinite may be said to possess most of the properties which we attach to the Finite. Two infinite aggregates, for example, can have an ordinal correspondence, and infinite aggregates submit, like finite ones, to arithmetical processes.

The mathematician analyses still more closely the relation between the Finite and the Infinite, as follows :—

He starts from the aggregate, which he analyses into the Finite and the Infinite, and the latter he analyses into the Transfinite and the Absolute. Of these two elements, one only has till just lately been the subject of mathematical treatment—the one called the Transfinite. It is the transfinite subdivision of the Infinite to which the idea of number is applicable, and which is, therefore, in a sense inseparable from the Finite. Infinite numbers or series ought then to be more correctly described as *Transfinite.* But the processes of mathematics do not end here; they reach up to the idea of the *Absolute Infinite,* the conception of which has been attained in recent years by mathematical work. The

results of this work may now be briefly summarised.

I. The Absolute appears to have the same relation to the Transfinite as the Transfinite to the Finite. If the Finite deals with numbers, and the Transfinite with series of numbers, the Absolute deals with series of series. Thus there are at least two examples of the Infinite within our grasp which lead up to the idea of the Absolute. One is the class of all classes of propositions; the other is the series of all worlds of thought, in Dedekind's sense.

II. The Finite, Transfinite, and Absolute can be further defined in this way. There is no greatest finite number, but there is a least transfinite number, which has been called Aleph 0, and which can be proved to be greater than any possible finite number, however large, because if there were a last number it must be smaller than the sum of the whole series. There are unending series of Alephs or infinite numbers, which are as distinct from one another in idea as 1 is from 0, and which can no more be derived from one another by a mathematical

process than 1 can be derived from 0, but can be reached in the same way by induction. Beyond the Transfinite we cannot discover in the Absolute the idea of least or of greatest.

III. The relation of cardinal and ordinal number also throws some light on the Finite, Transfinite, and Absolute. In the Finite, cardinals and ordinals are parallel to one another; in the Transfinite they strikingly diverge; in the Absolute we cannot trace any connection between cardinals and ordinals, *i.e.*, it is possible to have an ordinal series to which there can be no corresponding cardinal number or type.[1]

IV. If arithmetical processes are applied to the Finite, Transfinite, or Absolute, we get interesting results. We know the effect of addition, multiplication, and raising to a power, on the Finite. The first two processes have been applied to the Alephs; the last has been formulated, but the mathematical results have not yet been brought to a satisfactory con-

[1] Two transfinite aggregates can have an ordinal correspondence with one another.

clusion. Broadly speaking, we may say that the raising of an Aleph to a power may make it transcend the Finite and the Transfinite and melt into the Absolute. Thus all mathematical processes which find their goal in the Absolute would find their annihilation there. No finite mathematical conception would be applicable to it.

Now the conception of this Absolute Infinite, of which the aggregate of all ordinal numbers is perhaps a symbol,[1] has been subjected to criticism. Some mathematicians[2] think that it exists, but has no number. It is discovered by a logical process, but defies analysis and the application to it of the notion of number. All mathematical conceptions find in it their aim and conclusion. The importance of this theory, its practical importance, lies in the very much simpler mathematical formulæ that can be produced now that the logical process is shown to extend from the Finite to the Absolute Infinite (in the same way that the labour of summing a series arithmetically

[1] See G. Cantor, *Zur Lehre vom Transfiniten.* 1890.
[2] *e.g.* Mr. P. Jourdain, *Philosophical Magazine.* 1904.

by statement and addition is shortened by the application of algebraical principles which depend on larger knowledge). Its philosophical importance is great: the Absolute is here, as elsewhere, the goal of human thought, and is the mathematician's name for the highest power discoverable by human reason.

It would be very interesting to discuss the probable attitude of a Pascal or a Hegel to these mathematical conceptions, if they had been aware of them. Take Pascal's puzzle of the Finite and the Infinite. He thought that if the Finite could be subtracted from the Infinite, the Infinite would thereby lose some of its quality of infinity. How differently would it have appeared to him had he realised that an aggregate infinite cardinal can have subtracted from it either finite or transfinite terms: if transfinite terms, many different answers result, giving different degrees of transfinity: if only finite terms are taken away, the Infinite remains in its entirety.

How, again, would Hegel have rejoiced in a definition of thought and existence which

would bridge over the logical gulf in his system! Hegel asserted that thought and existence were one. He is objected to by many philosophers, who ask where is the *tertium quid* which makes it possible to reach from one to the other, or predicate their essential unity? But the mathematician defines existence as something which is not self-contradictory. Thought, then, to him is a form of existence, for thought is not self-contradictory; but the two, thought and existence, are not necessarily conterminous.[1] Hence, to say that non-contradiction is a fundamental condition of true thinking is as much as to say that it is a fundamental characteristic of real existence, and he identifies thought with reality.

Dr. Caird remarks that the secular conscience conceives of the Infinite as opposed to the Finite; the religious conscience treats the Infinite as real, presupposed by the illusory Finite. Where does the truth lie? Mathematics does not admit the necessity of adopt-

[1] The same result is hinted at by Mr. Taylor. Taylor, "Elements of Metaphysics," p. 22.

ing either view at the expense of the other.

Metaphysics standing alone produces results that may be disproved, but cannot be proved. Mathematics standing alone produces results that are susceptible of proof. Both are based on logic, and rest on the prerequisites of thought. Together they are a field for the best powers of human reason: metaphysics supplies insight, intuition, imagination; mathematics offers the indubitable proof and translates the ideal into the actual.

But the element in philosophical thought which, employing the psychological method, tends to the discussion of a theory of being rather than that of knowledge, and thus to the realisation of an ethical system rather than to metaphysical discovery, is averse from accepting these conclusions. It remains, therefore, for us to examine the criticism offered by the psychological school on what they call the mathematising of philosophy; and it will be found that the attack deals both with the ground of the alliance and its results.

A typical exponent of this school is Moisant,

who, in the *Revue Philosophique* for January 1905, attacked what he considered to be the characteristic of modern philosophy and also its vice. It will be observed that at the outset he reverses the *rôles* of philosophy and mathematics as we have apprehended them. Philosophy, he says, should expect to be inspired by mathematics, but should avoid its method. Next, he connects the modern movement with the theories of Leibniz, who aimed at substituting general formulæ for elementary forms of reason and calculation. These short cuts, which seem to the mathematician to liberate the mind from a burden which prevents it from employing its full activity, seem to the psychologist to tend to a mechanical method, in which the thinker is only aware of premises and results, and in which the mathematical concept tends to replace the real idea. Then he attacks the new definition of mathematics as the science of relations, asserting that it still contains notions of space.[1]

[1] Linear order, 1, 2, 3, &c. Circular. A CD B, A CD B. . . . The latter, it is true, involves the idea of separation. But this idea can be developed from those of inclusion and exclusion, which belong to the fundamental laws of thought.

Finally, he comes to the real question at issue, and enters into the comparison of a metaphysical and a mathematical problem. He takes as his subject the argument from the known to the unknown. Descartes had said that argument should lead from the known to the unknown, simple to complex, and had defined the first as that which could be known without the help of the second. This logical order of reasoning has been attributed to mathematics, but has been considered to be inapplicable to philosophy. Mathematics, in its recent development, by the argument from the Finite to the Infinite and back again, starts from two propositions, neither of which can be said to be axiomatic, because each in turn can be proved from the other, but in the course of argument from either mathematics makes use of the logical process. The real axiom, as has been shown, is that of *existence* or *being*. A metaphysical argument has the same root—that of existence—but a metaphysical problem deals with paradoxes, with questions which are sometimes defined as

having two answers, each equally correct, and sometimes as yielding no answer at all. The method of thesis, antithesis, and synthesis is in the Hegelian logic applied to their solution.

A mathematical and a metaphysical problem are not, then, problems of the same kind to be solved by the same method; nor is the conception of the mathematical Absolute reached in the same way as that of the metaphysical Absolute. We are even unable to say how far they correspond except in respect of their absoluteness.[1] But the contention of the mathematician to-day and of the epistemologist school of philosophy is not the identity of methods and results in the two sciences. It is the axiom of existence on which they both depend: the law of thought by which all methods are developed, and, above all, the *correlative value of each science to the other*, which allows us, in

[1] The Absolute, according to a recent metaphysical thinker, is "a conscious life which embraces the totality of existence, all at once, and in a perfect systematic unity, as the content of its experience."—Taylor, "Elements of Metaphysics," p. 60.

developing our knowledge from the standpoint of the two sciences, to recognise something of the greatness of the Absolute principle to which they both reach up, and in which their being consists.

II

PRAGMATISM AND A THEORY OF KNOWLEDGE

THE question before us is the relation of Pragmatism to a body of knowledge.

(*a*) One question at issue between the Idealist[1] and the Pragmatist has to do with the way in which each defines knowledge and gauges its ultimate aim. Both say that knowledge is relative, but one school asserts that the human mind slowly and laboriously uncovers or discovers what Goethe calls the "Living garment of Deity," *i.e.* the world of nature, and comes into a heritage of scientific truth which increasingly corresponds to the subject of his faith; the other claims that we live in a self-evolving universe in which in the course of long ages a new heaven and

[1] This word is used here in the most general and inclusive sense as applying to all thinkers who accept the reality of relations as part of a higher Unity.

a new earth may be created which are not foreseen or implied in present conditions. In other words, the Idealist finds the Divine in human life; he finds in his own small corner of the universe the microcosm and symbol of Infinity: the Pragmatist considers that nothing *is* which is not a result of human action, and lowers the Divine element to the result of individual human activity. A compromise between the two ideas on new and interesting lines has recently been made by Bergson. The Christian doctrine of Immanence and Transcendence also combines them.

Now the increase of a body of knowledge would seem to depend on the comparison of the successful working out of hypotheses with the discrepancies from theory that from time to time appear. Taken together, proofs and discrepancies point to the evidence of a larger law. This is Hegel's logic, and the principle, so far as it is here implied, is not denied in modern times, for no one wishes to found a logic on a study of discrepancies as such. Even W. James says, "Whenever you once place yourself at the point of view of any higher syn-

thesis you see exactly how it does, in a fashion, take up opposites into itself."[1] In fact, without the notion of unity, that of discrepancy could not exist: there must be a background on which the differences appear. The ultimate unity is symbolised in the Idealist doctrine of an Absolute.

The Absolute of Idealistic thought is not, however, now conceived of (as the Pragmatist would have us believe) as an abstract unity, but as one involving a social bond, and hence relations which can be described as personal, if we remember that the Personality of the Absolute transcends our notion of human personality. Such a conception of the term Absolute, a new reading of the theory of the One and the Many, has been led up to by Bradley and Royce by methods of logic, and without any reference to dogma. It has been conveniently expressed by Taylor. The argument is briefly that ultimate Reality must be One, Many, and Personal.

"For our conclusion that mere truth cannot be the same thing as ultimate reality was itself

[1] "A Pluralistic Universe," p. 99.

based upon the principle that only harmonious individuality is finally real, and this is the very principle employed by the intellect itself whenever it judges one thought-construction relatively higher or truer than another."[1]

And again:—

"If we speak of existence as a society, then we must be careful to remember that the individual unity of a society is just as real a fact of experience as the individual unity of the members which compose it, and that when we call the Absolute a society rather than a self, we do not do so with any intention of casting doubt upon its complete spiritual unity as an individual experience."[2]

The Absolute has been stated in modern thought to be One, Many, Real, and Personal or Social, and these terms of its qualification have been successively arrived at.

W. James's words ring hollow when he attempts to dissociate such a conception from the reality of which it is the crown and inclusive symbol, and type and essence. "I

[1] Taylor, "Elements of Metaphysics," p. 312.
[2] *Ibid.*, p. 350.

personally," he says, "give up the Absolute. I find it entangles me in metaphysical paradoxes that are inacceptable." He allows that there may be a God, though limited in power and goodness, "one helper amongst others, *primus inter pares* in the midst of all the shapers of the great world's fate." In such a system, as H. Jones has pointed out, "there is neither in the universe nor in God any principle to inspire or guide, or in any way to bring about the amelioration desired. The process is guided by no end. The universe begins by being an aggregate of accidents, pluralistic, discontinuous, irrational, and, of itself, cannot become otherwise. There is nothing actual within to change its character. . . . God is himself finite, helpless to bring about this great change, a part, and no more, of a universe broken in fragments."

Another form, and a very scholarly one, of the argument against the existence of an Absolute has been stated by Bax in the "Roots of Reality." He appears to have reached the conclusion that the *telos*, the goal of human thought, is not an Absolute

involving any notion of fixity, but that it may be conceived of as a "moving synthesis." He argues that everything of which we are conscious in the universe is seen against a background which itself moves, and is only realisable or distinguishable if it shifts upon something relatively motionless behind it. He concludes, therefore, that by analogy there is no Absolute, since what we perceive always implies something against which we perceive it; thus that there is no goal by which and at which the spirit of man can find rest. On his theory we could never claim to reach the conception of an Absolute, though he admits the progressive character of human thought, and the increasing reach, lucidity, and depth of the human mind. The true answer to this argument is that it proves exactly what it sets out to disprove. As it is acknowledged that only the permanent or the relatively permanent can produce the phenomena of change, so *the appearance of the goal of thought as a moving synthesis would presuppose an Absolute as a ground reality.*[1]

[1] A succession of what is disconnected is not change. Change is a succession within an identity: if not within the identity,

If in truth we were able to apprehend entirely the source of all life and the background of all experience, we might say that it did not exist for us *as an Absolute*, but the fact that whatever we perceive postulates an un ending series behind it, carries with it the proof of an Absolute Infinite. (This conclusion is led up to by the mathematician's idea of the series of all finite and transfinite ordinal numbers.) Some part of this argument has been already suggested in Ormond's "Foundations of Knowledge," and so far was used by Mr. Illingworth in the "Doctrine of the Trinity."[1]

"From a deeper metaphysical point of view it is the concept of evolution itself that must submit to the determination of knowledge, for it will be found that in so far as it becomes epistemologically necessary to ground relative processes in an Absolute experience, just so far will it become necessary also to connect the evolutionary aspect of the world itself with

there is no change, only analysis and re-grouping. The closer our knowledge is of ourselves or anything else, the more we see that *change is the expression in time of an identity.*

[1] Illingworth, "The Doctrine of the Trinity," p. 6.

a ground reality that is stable, and involves the flux of change only as transcending and including it."[1]

The further answer that any judgment, even the Pragmatist's "judgment of value," implies an Absolute has been stated in his Oxford Lectures[2] by Professor H. Jones.

(*b*) The next point we should like to work out is the relation of fact to law. The Pragmatist denies scientific law and also logic, and makes his appeal to facts. No conclusion can be drawn from that denial except by the use of logic itself. If he consistently denied logic, his position would be unassailable by logic, but he uses the method he denies, and is thus open to attack. On the subject of the Laws of Science the Pragmatist points out truly that there is no actual continuity between a fact and a law. But laws are concepts, the result of mental activities; they are themselves subject to the laws of logic. "They were means, and you make them ends," complains the Pragmatist. That is just what nature

[1] Ormond, "Foundations of Knowledge," p. 19.
[2] 1908–10.

herself does. She perfects means, such as the means of supporting life, and then these become ends. Language, again, is at first a means, and then becomes an end. So does any science change its character to the on-looker. A law, too, though it generalises facts, is a limit on absolute generalisation. It thus stands midway between the abstraction and the fact. The Pragmatist, however, opposes to law what he calls a new fact—what should rather be called a hypothesis. He asserts that in every event, action, experiment, there is a margin unseen and unrecognised by us; that at every moment, therefore, the unknown, the unexpected, may take shape and voice and denounce all our careful and reasoned conclusions. "Why should the sun rise to-morrow because he has risen to-day and yesterday?" asks the Pragmatist. "We are making an enormous assumption," he says, "in claiming the uniformity of Nature and the principle of causality." The Idealist answers that the Pragmatist makes a larger assumption in doubting the truth of the principles, which though relative and not absolute, still do work

out in practice, than the Idealist does in his act of faith. In fact, the act of faith is rational as well as natural; it is the act of doubting that is in this case due to a mere scholastic quibble. It is the Idealist and not the Pragmatist who makes his appeal to the truth of facts. Each day that the sun goes on rising finds the Idealist in a better philosophical position and the Pragmatist in a worse, except on the assumption that the link between man and the external world is a false imagination. Let us emphasise:—It is the Pragmatist who quibbles with logic, and the Idealist who appeals to facts.

(*c*) Now there are certain facts and certain deductions from facts, well known to mathematicians, which we should like to quote here as having a bearing on the theory of the Absolute, because they deal with aspects of Infinity, and mark a connection between the world as we know it and the concepts of the philosopher. All have the support of science, and furnish the Idealist philosopher with examples which support his theories, and strengthen his position in the face of the

Pragmatist attack. They have to do with the theory of Infinity as shown in :—

I. The Indefinite Regress.

II. Infinite series.

III. Dimensions in space and time.

Before entering upon them we must repeat that the question of number and series in mathematics is independent of the assumptions of space and time. As a science, mathematics could exist outside them : order is not necessarily spatial or temporal. Our conclusions, therefore, cannot be attacked on the ground that they are based on Euclidean conceptions of space : they are based on the laws of logic.

I. The Indefinite Regress

Hume and, later, Kant argued that by the principle of association when we think of one quality of a thing the others are naturally brought before our minds, and thus that we get into the habit of attributing to the notion of the thing a certain group of qualities. And it is true that we do attend to a thing all at

once, including in the notion of it all the qualities which we know belong to it.

Now experience, according to Leibniz, gives us an example of a unity which embraces a multiplicity of detail. Thus a thing is one substance as embodying an individual experience, and its qualities belong to it in the same sense as the constituents of experience belong to the single experience. These qualities are in relation. (The Pragmatist denies the existence of relations as part of a higher unity.[1]) But they are not only relation, since relation always implies something more than itself. Let us take the example of number. Numbers could never have been counted if there had not been things to count. Now suppose each quality could be analysed into a new relation, we should still not get rid of the quality. At each stage there remains a quality in relation, and this goes on to Infinity. Such a constant subdivision perhaps results from our finite experience seizing facts in a disjointed way. When we analyse a law in its working, we

[1] James, "A Pluralistic Universe," p. 80.

always do seem to come to this Indefinite Regress. Now it has been the reproach against metaphysics, as uttered by the Pragmatist, that there is no correspondence in scientific fact to this road into Infinity.

W. James asserts: "But in point of fact, nature doesn't make eggs by making first half an egg, then a quarter, then an eighth, &c., and adding them together. She either makes a whole egg at once or none at all, and so of all her other units. It is only in the sphere of change, then, where one phase of a thing must needs come into being before another phase can come, that Zeno's paradox gives trouble. And it gives trouble then only if the successive steps of change be infinitely divisible."[1]

The sphere of change, however, one would answer, includes all nature, and science in its discoveries acts on the hypothesis that these steps of change may be infinitely divisible. Royce held to it firmly that any consistent attempt to make an orderly arrangement of the terms of an infinite whole must lead to the Inde-

[1] "A Pluralistic Universe," p. 230.

finite Regress. And he further shows the connection with the fact that an infinite series can be adequately represented by a part of itself.

In the Boyle Lecture, delivered in Oxford in 1908, on the properties of radium, two facts emerged which show that the Indefinite Regress is now recognised in science.

First, that in the region of experiment we become aware of groups of elements allied to radium, which seem, in the number of individuals in their groups, to follow a simple arithmetical progression.

Secondly, that radio-active elements lose in activity at a certain rate, which always represents an exact proportion of the mass which remains. The tremendous disintegrating force slackens in exact relation to the time which passes, so that the smaller the morsel the less the relative loss of mass. Here, then, is the Indefinite Regress. In the world of fact as well as of ideas we are dealing with aspects of Infinity.[1]

[1] See A. T. Cameron, "Radio-Chemistry," p. 17: "The curves illustrate two further points. They approach constant value towards the end of a month, but it is seen that they reach a final value only at infinite time. This property is common

II. Infinite Series

There are other aspects of Infinity which we can get at by studying series, and which in the conception of series of series give strength and point to the philosophic conception of an Absolute.

Prof. C. Keyser develops this thought, and shows (in two recent articles, January and April 1909, in the *Hibbert Journal*) that certain theological dogmas, such as the doctrine of the Trinity, and certain attributes of the Divine Being, such as Omniscience and Omnipresence, are entirely conceivable by the human mind if regarded without the paralysing limitations of the Finite. He shows that in our mathematical formulæ which have to do with infinite series we have the exact replica of what to the lay, non-mathematical mind seem to be the paradoxes of the Athanasian Creed. He first shows that in a mathematical analogy points of view about an Infinite Being, even if

to all such curves; it illustrates the fact that the *life* of a radio-active element is infinite." It is explained in the same book (p. 31) that "infinity is only a relative term; in this connection it only means a longer time than we can measure."

partially discordant, may all be true if regard is had to His Infinity.[1]

Further, he shows that certain assumptions, such as *the whole is greater than its part*, are inapplicable to Infinite Being. The conception of a Trinity in Unity in which "none is afore or after other, none is greater or less than another, but the whole three persons are co-eternal together and co-equal" is rationally conceivable by the mathematician who is familiar with the theory of manifolds.[2]

We have, he shows, three infinite manifolds: —

E of the even integers.

O of the odd ones.

F of the fractions having integers for their terms.

No two of these have a single element in common, yet the three together constitute one

[1] His theology is not so good as his mathematics; he seems to think that in the Creed we assert our belief in the Incomprehensible, in the sense of that which is "not capable of being seized by the mind," instead of in that which is "untrammelled by limitations." The word is *immensus*, best translated infinite.

[2] *Hibbert Journal*, 1909, pp. 626–28.

manifold M, that is exactly equal in wealth of elements to *each* of its infinite components.

Again, there is the apparent opposition between the Omniscience of God and the freedom of man. The antithesis disappears if we realise that from the point of view of Infinites the dignity and power of Omniscience remain the same, even if some part of experience is not yet drawn into the sphere of Omniscience.[1]

Here we have the present conceived of as a moving plane separating the unknown from the known. The "past" can be said to be known, though its content changes every instant. This is the real answer to W. James's cry that he could accept an Absolute if it had even the fragment of an "other." There can be this "other," and yet the Absolute still remains an Absolute.

The doctrine of *Omnipresence* follows from the argument of the Continuum (which is the aggregate of all real numbers). Thus the number of points in space of infinite dimension is no greater than the number of points in any part of space as known to us. The whole

[1] *Hibbert Journal*, 1909, p. 629.

is incarnate in every part, because to each part, in however small an atom, corresponds a point in the universal whole, and the number of points in a space of infinite dimensions is equal to the number of points in a straight line however small.

And this is true not merely of points but also of forces. "The Universe is dynamic, charged throughout with innumerable modes of motion. Each point, however, of any moving thing—an ion of gas, a vibrating fibre of brain—is represented by a corresponding point in S (a small typical atom), and so within the tiny sphere—indeed, in every room, however small—the whole dynamics of the universe is depicted completely and co-enacted by motion of points and transformation of point configurations. There in miniature proceed at once the countless play and interplay of every kind of motion, small and large, simple and complex, the quivering dance of the molecule, the wave and swing of universal æther."[1]

[1] *Hibbert Journal*, 1909, p. 632.

III. Dimensions in Space

There is another argument, one relating to the theories of time and space, which greatly affects the conception of omnipresence. This is the argument of the many dimensions, called by Keyser the "radiant concept of hyper-space, which only a generation ago was regarded, even by mathematicians—most adventurous of men —as being purposeless and vain, but which meanwhile has advanced so rapidly to commanding position that even the following statement by Poincaré, in his recent address before the International Mathematical Congress at Rome on 'L'Avenir des Mathématiques,' is well within the limits of conservatism: 'Nous sommes aujourd'hui tellement familiarisés avec cette notion que nous pouvons en parler, même dans un cours d'université, sans provoquer trop d'étonnement.' The fact is that the doctrine already exists in a vast and rapidly growing literature, flourishes in all the scientific languages of the world, and in its essential principles has become for mathematics as orthodox as the multiplication table."

The present position of the theory is briefly this: If there did not exist a fourth dimension, we could not be aware of a third as such, and so on. Are we then looking out upon a third dimensional world, and realising it as such because we are mentally capable of conceiving dimensions beyond it? Our world sensibly contains one dimensional and two dimensional facts—the first such as a time series, for which one number is sufficient to fix a point, and the second such as a plane where position can be fixed by two numbers. Does our world contain facts of other dimensions?

"All particles of air are four-dimensional in magnitude when, in addition to their position in space, we also consider the variable densities which they assume, as they are expressed by the different heights of the barometer in the different parts of the atmosphere. Similarly all conceivable spheres in space are four-dimensional magnitudes, for their centres form a three-dimensional point-aggregate, and around each centre a one-dimensional totality of spheres, the radii of which can be expressed by every numerical magnitude from zero to

infinity. Further, if we imagine a measuring-stick of invariable length to assume every conceivable position in space, the positions so obtained will constitute a five-dimensional aggregate. For in the first place one of the extremities of the measuring-stick may be conceived to assume a position at every point of space, and this determines for one extremity alone of the stick a three-dimensional totality of position, and, secondly, as we have seen above, there proceeds from every such position of this extremity a two-dimensional totality of directions, and by conceiving the measuring-stick to be placed lengthwise in every one of these directions, we shall obtain all the conceivable positions which the second extremity can assume, and consequently the dimensions must be $3+2$ or $5 \ldots$" &c., &c.[1]

Mathematicians have for long done problems in the seventh and eighth dimensions. They have told us that you cannot tie a knot in the second dimension, because there is no up or down, and the threads would not cross—nor in

[1] Schubert, "Mathematical Essays and Recreations," pp. 70, 71.

the fourth, because the knot would pull out in a new direction and would not hold. But it has only lately been realised that fourth and other dimensions may be actual fact in the world round us. Of course, from the point of view of a point there are only three dimensions to be known, but to a line in the same space there are five, to the surface probably six. Our intelligence at present does not go beyond the point; but if we could think of space from the point of view of a solid, worlds upon worlds would rise before our view.

Of the fourth dimension we can discover some facts by analogy. We can count the edges of its typical figure, and apply thought to determining some of its conditions. But a more interesting subject of research is the inquiry into the light thrown by the theory of four dimensions on the determination of certain atoms in chemistry, that are known to be distinct elements, but could only be determined actually in another dimension.[1]

[1] See Van t' Hoff, *La Chimie dans l'espace*, and Schubert, "Mathematical Essays and Recreations," pp. 88–89.

"In chemistry, the molecules of a compound body are said to consist of the atoms of the elements which are contained in the body, and these are supposed to be situated at certain distances from one another and to be held in their relative positions by certain forces. At first the centres of the atoms were conceived to lie in one and the same plane. But Wislicenus was led by researches in paralactic acid to explain the differences of isomeric molecules of the same structural formulæ by the different positions of the atoms in *space*. In fact, four points can always be so arranged in space that every two of them may have any distance from each other; and the change of one of the six distances does not necessarily involve the alteration of any other.

"But suppose our molecule consists of five atoms? Four of these may be so placed that the distance between any two of them can be made what we please. But it is no longer possible to give the fifth atom a position such that each of the four distances by which it is separated from the other atoms may be what we please. On the contrary, the fourth dis-

tance is dependent on the three remaining distances, for the space of experience has only three dimensions. If, therefore, I have a molecule which consists of five atoms, I cannot alter the distance between two of them without at least altering some second distance. But if we imagine the centres of the atoms placed in a four-dimensioned space, this can be done; all the ten distances which may be conceived to exist between the five points will then be independent of one another. To reach the same result in the case of six atoms we must assume a five-dimensional space, and so on."[1]

Here we see that if chemistry as a science is bound to take account of all its facts, the scientist is confronted with a problem of dimensions that is really a problem of Infinity applied not, as in the other cases quoted, to number, but to space.

And there is a reason which explains why the same problem tends to appear in these different ways. Both time and space can be

[1] Schubert, "Mathematical Essays and Recreations," pp. 88–89. See also Mach, "Conservation of Energy" (trans. Open Court Publishing Co.).

most correctly thought of as *series:* the former known to us as possessing one direction, though possibly involving more, and the latter three, though possibly involving more. Time is not a thing nor a condition, but it is the way in which we are enabled to apprehend the relations of actions to one another. The assumption of the Pragmatist, that a different date in history is a new condition which might affect a chemical experiment, is meaningless, unless by that he intends to say that at the different date new conditions prevailed.

The general conclusion of recent thought is then to establish the Idealist position more strongly by an appeal to mathematical argument. This argument is strengthened by finding at the present time some support in scientific fact and experiment. The Idealist therefore appeals to fact, and his position rests ultimately on a truth which has its aspects of conformity with scientific experiment and with logical or mathematical proof.

Printed by BALLANTYNE, HANSON & Co.
Edinburgh & London